

Pseudo-Kähler Lie algebras with abelian complex structures

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March 8, 2013

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Abstract

We study Lie algebras endowed with an abelian complex structure which admit a symplectic form compatible with the complex structure. We prove that each of those Lie algebras is completely determined by a pair (U, H) where U is a complex commutative associative algebra and H is a sesquilinear hermitian form on U which verifies certain compatibility conditions with respect to the associative product on U . The Riemannian and Ricci curvatures of the associated pseudo-Kähler metric are studied and a characterization of those Lie algebras which are Einstein but not Ricci flat is given. It is seen that all pseudo-Kähler Lie algebras can be inductively described by a certain method of double extensions applied to the associated complex associative commutative algebras.

1 Introduction

An abelian complex structure on a Lie algebra \mathfrak{g} is a linear map $GL(\mathfrak{g})$ such that $J^2 = -\text{id}_{\mathfrak{g}}$ and $[Jx, Jy] = [x, y]$ for all $x, y \in \mathfrak{g}$. It is obvious that the Nijenhuis tensor for such a J vanishes and, therefore, the structure is integrable. The name *abelian* comes from the fact that the eigenspaces of J in the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} are abelian Lie algebras. Note that they are a particular case of the so called nilpotent complex structures [11].

A pseudo-Kähler Lie algebra is a symplectic Lie algebra (\mathfrak{g}, ω) with a complex structure J which is skew-symmetric with respect to ω . Such a Lie algebra is equipped with the pseudo-Riemannian metric g defined by $g(x, y) = \omega(Jx, y)$ for all $x, y \in \mathfrak{g}$ for which the complex structure is parallel. The aim of this paper is to study the structure and main properties of Lie algebras which are pseudo-Kähler for an abelian complex structure.

If an algebra \mathfrak{g} admits an abelian complex structure it must be 2-step solvable. One can easily verify that Lie algebras of dimension 2 have an abelian complex structure (actually, all complex structures on these algebras are abelian) and that every non-degenerate skew-symmetric bilinear form ω is a symplectic form compatible with the complex structure. So, 2-dimensional Lie algebras are the first examples of pseudo-Kähler Lie algebras with abelian complex structure. Recall that the unique non abelian 2-dimensional Lie algebra is the algebra $\mathfrak{aff}(\mathbb{R})$ of affine motions of \mathbb{R} . An important role in the sequel will be

played by the Lie algebras $\mathfrak{aff}(\mathcal{A})$ constructed as the tensor product $\mathfrak{aff}(\mathcal{A}) = \mathfrak{aff}(\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{A}$ of $\mathfrak{aff}(\mathbb{R})$ with an associative commutative algebra \mathcal{A} . A well-known example of these algebras is the underlying Lie algebra of the Kodaira-Thurston manifold, which is the algebra $\mathfrak{aff}(\mathcal{A})$ when \mathcal{A} is the 2-dimensional nilpotent power algebra. All the algebras $\mathfrak{aff}(\mathcal{A})$ carry abelian complex structures but in order to admit a compatible symplectic form the algebra \mathcal{A} must verify further conditions. The existence of a symmetric bilinear form B on \mathcal{A} such that the pair (\mathcal{A}, B) is a symmetric algebra [12] guarantees that the corresponding Lie algebra $\mathfrak{aff}(\mathcal{A})$ admits a pseudo-Kähler structure for an abelian complex structure; however, these examples do not exhaust the family of pseudo-Kähler Lie algebras with abelian complex structure.

In this paper we will show that every pseudo-Kähler Lie algebra with abelian complex structure is completely characterized by a pair (U, H) where U is a complex associative commutative algebra and H a sesquilinear hermitian form on U which verifies certain compatibility conditions. This characterization let us calculate some nice formulas for the Riemannian and the Ricci curvatures of the pseudo-Kähler metric and lead us to a complete description those algebras in our family for which the metric is Einstein but not Ricci flat. Moreover, in the last section we prove that all the pseudo-Kähler Lie algebras with abelian complex structure can be obtained by successive application of a method of double extension, which consist of a central extension and a generalized semi-direct product, on the associated pairs (U, H) starting from an algebra $\mathfrak{aff}(\mathcal{A})$.

2 Preliminaries

We first recall some basic definitions [2], [3], [15], [16]. All the algebras in the paper are considered real or complex finite dimensional algebras.

Definition 2.1 A *complex structure* on a real Lie algebra \mathfrak{g} is a linear map $J \in \mathfrak{gl}(\mathfrak{g})$ such that $J^2 = -\text{id}_{\mathfrak{g}}$ and

$$[Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy] \quad \text{for all } x, y \in \mathfrak{g}.$$

Two Lie algebras endowed with complex structures $(\mathfrak{g}_1, J_1), (\mathfrak{g}_2, J_2)$ are said to be *holomorphically equivalent* if there is an isomorphism $\psi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $J_1 = \psi^{-1} \circ J_2 \circ \psi$.

We will say that a complex structure J is *abelian* if $[Jx, Jy] = [x, y]$ holds for all $x, y \in \mathfrak{g}$.

Definition 2.2 Let \mathfrak{g} be a complex or a real Lie algebra. A *product structure* on \mathfrak{g} is a linear map $K \in \mathfrak{gl}(\mathfrak{g})$ such that $K^2 = \text{id}_{\mathfrak{g}}$, $K \neq \text{id}_{\mathfrak{g}}$ and

$$[x, y] = K[Kx, y] + K[x, Ky] - [Kx, Ky] \quad \text{for all } x, y \in \mathfrak{g}.$$

The product structure K provides a decomposition of the vector space \mathfrak{g} as the direct sum of the eigenspaces $\mathfrak{g}_+ = \ker(K - \text{id}_{\mathfrak{g}})$, $\mathfrak{g}_- = \ker(K + \text{id}_{\mathfrak{g}})$. Actually, \mathfrak{g}_+ and \mathfrak{g}_- are subalgebras of \mathfrak{g} . When both subalgebras \mathfrak{g}_{\pm} have the same dimension, the product structure is said to be a *paracomplex structure* on \mathfrak{g} .

We will say that a product structure K is *abelian* if $[Kx, Ky] = -[x, y]$ holds for all $x, y \in \mathfrak{g}$. In such case one easily verifies that the subalgebras \mathfrak{g}_{\pm} are actually abelian.

Remark 2.1 One can also use Definition 2.1 to define the notion of complex structure for complex Lie algebras, considering J to be \mathbb{C} -linear. However, this is not very interesting since for a complex Lie algebra \mathfrak{g} there is one-to-one correspondence between such \mathbb{C} -linear complex structures and product structures on \mathfrak{g} . Actually, a straightforward computation shows that a \mathbb{C} -linear complex mapping J on a complex

Lie algebra \mathfrak{g} is a complex structure if and only if the map $K = iJ$ is a product structure, where i stands for the imaginary unit. Further, J is abelian if and only if K is so.

The following result is very well-known (see, for instance, [3]):

Lemma 2.1 *Every Lie algebra admitting an abelian complex structure is 2-step solvable.*

Examples 2.1 Some interesting examples of Lie algebras admitting abelian complex structures are:

1. *The Lie algebra $\mathfrak{aff}(\mathcal{A})$ of a commutative associative algebra.*

Let \mathcal{A} be a commutative associative algebra. The vector space $\mathcal{A} \oplus \mathcal{A}$ with the product defined by $[(a, b), (a', b')] := (0, ab' - a'b)$ for all $a, a', b, b' \in \mathcal{A}$, is a Lie algebra denoted by $\mathfrak{aff}(\mathcal{A})$ [8]. Note that, actually, $\mathfrak{aff}(\mathcal{A}) = \mathfrak{aff}(\mathbb{R}) \otimes \mathcal{A}$ with the bracket $[x \otimes a, y \otimes a'] = [x, y] \otimes aa'$. It is clear that the linear map J on $\mathfrak{aff}(\mathcal{A})$ defined by: $J(a, b) := (-b, a)$, for $a, b \in \mathcal{A}$, is an abelian complex structure on $\mathfrak{aff}(\mathcal{A})$.

2. *The underlying real algebra of a complex Lie algebra with an abelian product structure*

When (\mathfrak{g}, K) is a complex Lie algebra with a \mathbb{C} -linear abelian product structure, the underlying real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is naturally endowed with the abelian complex structure defined by the \mathbb{R} -linear map $J = iK$.

Definition 2.3 We say that a Lie algebra \mathfrak{g} admits a *symplectic structure* if it admits a non-degenerate scalar 2-cocycle ω . The pair (\mathfrak{g}, ω) is said to be a *symplectic Lie algebra*. Two symplectic Lie algebras $(\mathfrak{g}_1, \omega_1)$, $(\mathfrak{g}_2, \omega_2)$ are said to be *symplectomorphic* if there exists a Lie algebras homomorphism $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $\omega_1(x, y) = \omega_2(\varphi x, \varphi y)$ for all $x, y \in \mathfrak{g}_1$.

Remark 2.2 A symplectic Lie algebra (\mathfrak{g}, ω) is naturally endowed with a structure of left-symmetric algebra compatible with the Lie structure; this is to say, the product defined by $\omega(x \cdot y, z) = -\omega(y, [x, z])$ for $x, y, z \in \mathfrak{g}$ verifies $[x, y] = x \cdot y - y \cdot x$ and

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Geometrically, this means that each Lie group with Lie algebra \mathfrak{g} can be equipped with the torsion free flat left-invariant connection defined by $\nabla_x^\omega y = x \cdot y$ for $x, y \in \mathfrak{g}$.

Definition 2.4 Let (\mathfrak{g}, ω) be a symplectic Lie algebra and J a complex structure on \mathfrak{g} such that $\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in \mathfrak{g}$. The triple $(\mathfrak{g}, \omega, J)$ will be called a *pseudo-Kähler Lie algebra*. If the complex structure is abelian we will say that the algebra \mathfrak{g} is endowed with an *abelian pseudo-Kähler structure*. The pseudo-Riemannian metric g defined on \mathfrak{g} by $g(x, y) = \omega(Jx, y)$ will be called the *pseudo-Kähler metric* of $(\mathfrak{g}, \omega, J)$.

The following lemma will be useful in the next sections.

Lemma 2.2 *Let $(\mathfrak{g}, \omega, J)$ be a pseudo-Kähler Lie algebra where J is abelian. The left-symmetric product defined by ω on \mathfrak{g} verifies $J(x \cdot y) = (Jx) \cdot y$ for all $x, y \in \mathfrak{g}$.*

Proof. If $x, y, z \in \mathfrak{g}$ then we have

$$\omega(J(x \cdot y), z) = -\omega(x \cdot y, Jz) = \omega(y, [x, Jz]) = -\omega(y, [Jx, z]) = \omega((Jx) \cdot y, z),$$

which proves the result. \square

Remark 2.3 One may think that for every associative and commutative algebra \mathcal{A} the corresponding Lie algebra $\mathfrak{aff}(\mathcal{A})$ always admits an abelian pseudo-Kähler structure. However the result is not true and one may find associative commutative algebras \mathcal{A} such that $\mathfrak{aff}(\mathcal{A})$ does not even admit a symplectic form. For instance, if \mathcal{A} is the 3-dimensional associative algebra spanned by a_1, a_2, a_3 with non-trivial products $a_1 a_1 = a_2 a_2 = a_3$, then the algebra $\mathfrak{aff}(\mathcal{A})$ is spanned by E_1, \dots, E_6 where $E_j = (a_j, 0)$ and $E_{3+j} = (0, a_j)$ for $1 \leq j \leq 3$ and their only non-trivial brackets are $[E_1, E_4] = [E_2, E_5] = E_6$. If ω is a 2-cocycle on $\mathfrak{aff}(\mathcal{A})$, one has

$$\begin{aligned} \omega(E_6, E_j) &= \omega([E_1, E_4], E_j) = -\omega([E_4, E_j], E_1) - \omega([E_j, E_1], E_4) \\ \omega(E_6, E_j) &= \omega([E_2, E_5], E_j) = -\omega([E_5, E_j], E_2) - \omega([E_j, E_2], E_5). \end{aligned}$$

The first identity shows that $\omega(E_6, E_2) = \omega(E_6, E_3) = \omega(E_6, E_5) = 0$ and the second one implies $\omega(E_6, E_1) = \omega(E_6, E_4) = 0$. Therefore $\omega(E_6, x) = 0$ for all $x \in \mathfrak{aff}(\mathcal{A})$, which proves that the 2-cocycle cannot be non-degenerate.

We recall the following definition [12]:

Definition 2.5 An associative algebra \mathcal{A} over \mathbb{K} is said to be a *symmetric algebra* if \mathcal{A} is equipped with a non-degenerate symmetric bilinear form $B : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{K}$ that satisfies $B(ab, c) = B(a, bc)$ for $a, b, c \in \mathcal{A}$.

It should be noticed that, according to the results given in [9], one can construct a symmetric associative commutative algebra by applying the method of T^* -extension to an arbitrary associative commutative algebra.

Examples 2.2 The following examples of pseudo-Kähler Lie algebras with abelian complex structures show that the class of such algebras is a wide one.

1. *The Lie algebra $\mathfrak{aff}(\mathcal{A})$ of a symmetric associative commutative algebra.*

If (\mathcal{A}, B) is a commutative associative symmetric algebra then the algebra $\mathfrak{aff}(\mathcal{A})$ with the abelian complex structure defined above and the symplectic form given by

$$\omega((a, b), (a', b')) := B(a, b') - B(b, a'), \quad a, a', b, b' \in \mathcal{A}$$

is pseudo-Kähler. In this case we will say that $\mathfrak{aff}(\mathcal{A})$ is equipped with the *standard pseudo-Kähler structure*.

A simple calculation shows that the left-symmetric product defined by ω on $\mathfrak{aff}(\mathcal{A})$ is given by

$$(a, b) \cdot (a', b') = -(aa', ba'), \quad a, b, a', b' \in \mathcal{A},$$

which is, actually, associative.

2. *The underlying real algebra of a complex Lie algebra with an abelian para-Kähler structure*

If $(\mathfrak{g}, K, \omega)$ is a complex Lie algebra with an abelian para-Kähler structure, then the underlying real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is pseudo-Kähler if one considers the complex structure $J = iK$ and the symplectic form $\omega_{\mathbb{R}} = \text{Re}(\omega)$ where $\text{Re}(\omega)$ stands for the real part of ω .

It should be noticed that the real dimension of each of those algebras is $4k$ for some $k \in \mathbb{N}$.

3. *An example with non associative left-symmetric product*

Let us consider the 6-dimensional real Lie algebra \mathfrak{g} linearly spanned by $\{E_1, \dots, E_6\}$ with the non-trivial brackets

$$\begin{aligned} [E_1, E_2] &= -E_3 - \frac{1}{2}E_6, & [E_1, E_4] &= 4E_2, & [E_1, E_5] &= \frac{1}{2}E_3 + E_6, \\ [E_2, E_4] &= \frac{1}{2}E_3 + E_6, & [E_4, E_5] &= -E_3 - \frac{1}{2}E_6. \end{aligned}$$

Notice that \mathfrak{g} is the Lie algebra denoted by \mathfrak{n}_7 in [3]. The skew-symmetric bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ defined by

$$\omega(E_1, E_3) = 4, \quad \omega(E_1, E_4) = -2, \quad \omega(E_2, E_5) = -2, \quad \omega(E_4, E_6) = 4$$

and $\omega(E_j, E_k) = 0$ for the other cases in which $j < k$, turns to be a symplectic form on \mathfrak{g} and, further, the map $J \in \mathfrak{gl}(\mathfrak{g})$ such that $J(E_j) = E_{3+j}$, $J(E_{3+j}) = -E_j$, for $j = 1, 2, 3$, is an abelian complex structure compatible with ω . Therefore, $(\mathfrak{g}, \omega, J)$ is pseudo-Kähler.

However, the left-symmetric product defined by ω is not associative. Actually, one can easily verify that, for instance, $(E_1 \cdot E_4) \cdot E_1 \neq E_1 \cdot (E_4 \cdot E_1)$.

Note that this algebra provides an example of pseudo-Kähler Lie algebra with abelian complex structure which is not of the types of the two previous examples.

A complex structure J on a real Lie algebra \mathfrak{g} provides a decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$ as the direct sum of the vector spaces $\mathfrak{g}^{1,0} = \ker(J^{\mathbb{C}} - i\text{id}_{\mathfrak{g}})$, $\mathfrak{g}^{0,1} = \ker(J^{\mathbb{C}} + i\text{id}_{\mathfrak{g}})$, where $J^{\mathbb{C}}$ is the \mathbb{C} -linear map on $\mathfrak{g}^{\mathbb{C}}$ defined by $J^{\mathbb{C}}(x_1 + ix_2) = Jx_1 + iJx_2$ for $x_1, x_2 \in \mathfrak{g}$. One can easily verify that $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are complex subalgebras of $\mathfrak{g}^{\mathbb{C}}$ and, obviously, they have the same dimension. Hence, the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is naturally endowed with a paracomplex structure. Further, if the complex structure J is abelian, then the subalgebras $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are abelian and, thus, the paracomplex structure on $\mathfrak{g}^{\mathbb{C}}$ is also abelian [6]. Further, if the real Lie algebra is pseudo-Kähler, then its complexification results to be para-Kähler. The following result will be used in next section. We omit its proof since it is nothing but a straightforward calculation.

Lemma 2.3 *If $(\mathfrak{g}, \omega, J)$ be a pseudo-Kähler Lie algebra with abelian complex structure then its complexification $\mathfrak{g}^{\mathbb{C}}$ is naturally endowed with the abelian para-Kähler structure defined by the complex symplectic form $\omega^{\mathbb{C}}(x + iy, x' + iy') = \omega(x, x') - \omega(y, y') + i\omega(x, y') + i\omega(y, x')$, where $x, x', y, y' \in \mathfrak{g}$ and the abelian paracomplex structure $K(x + iy) = -Jy + iJx$. Further, the left-symmetric product defined by $\omega^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ is obtained from the left-symmetric product on \mathfrak{g} as follows:*

$$(x + iy) \cdot (x' + iy') = x \cdot x' - y \cdot y' + ix \cdot y' + iy \cdot x',$$

for $x, y, x', y' \in \mathfrak{g}$

In order to fix notations, we will recall some well-known definitions on linear algebra. In the sequel, for a complex number $\alpha \in \mathbb{C}$, we will denote by $\operatorname{Re}(\alpha)$ and $\operatorname{Im}(\alpha)$ respectively its real and imaginary part and $\bar{\alpha}$ will mean its complex conjugate.

Let V be a complex vector space. A *semilinear map* $\tau : V \rightarrow V$ is a \mathbb{R} -linear map such that $\tau(\alpha v) = \bar{\alpha}\tau(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$. A *sesquilinear form* is a \mathbb{R} -bilinear map $H : V \times V \rightarrow \mathbb{C}$ which is \mathbb{C} -linear in the left component and semilinear in the right one. Further, a sesquilinear form H is said to be *hermitian* if $H(v_1, v_2) = \overline{H(v_2, v_1)}$ holds for all $v_1, v_2 \in V$. Note that, even though we will only consider non-degenerate hermitian forms, we shall not impose H to be definite.

Let (V, H) be a complex vector space with a non-degenerate hermitian form. If W is a complex subspace of V , then we will denote by W^\perp its H -orthogonal subspace, this is to say

$$W^\perp = \{ x \in V : H(x, y) = 0 \text{ for all } y \in W \}.$$

We remind that if $F : V \rightarrow V$ is a \mathbb{C} -linear map, its H -adjoint map $F^* : V \rightarrow V$ is uniquely defined by $H(F^*x, y) = H(x, Fy)$ for $x, y \in V$.

Remark 2.4 All through the paper we will consider traces of \mathbb{R} -linear maps. Even for a \mathbb{C} -linear map, the trace considered will be the trace of the corresponding \mathbb{R} -linear map on the underlying real vector space.

A simple calculation shows that for a semilinear map τ on a complex vector space one always has $\operatorname{trace}(\tau) = 0$.

3 Compatible complex associative commutative algebras

As we will see below, every pseudo-Kähler Lie algebra with abelian complex structure is obtained from a pair (U, H) where U is a complex associative commutative algebra and H a non-degenerate hermitian form on U which verifies some compatibility conditions with respect to the associative product on U . We start this section with the construction of the associated pseudo-Kähler Lie algebras for a given pair (U, H) .

Definition 3.1 We will say that a complex associative commutative algebra U endowed with a non-degenerate hermitian sesquilinear form H is *compatible with an abelian pseudo-Kähler structure* or, shortly, *APK-compatible* if and only if for every $x, y, z \in U$ it holds that

$$xR_z^*y = yR_z^*x, \tag{1}$$

where R_z stands for the multiplication by z in U .

Proposition 3.1 *Let U be a complex associative commutative algebra and H a non-degenerate hermitian form such that the pair (U, H) is APK-compatible.*

Let \mathfrak{g}_U denote the underlying real vector space of U and define $[\cdot, \cdot] : \mathfrak{g}_U \times \mathfrak{g}_U \rightarrow \mathfrak{g}_U$, $J : \mathfrak{g}_U \rightarrow \mathfrak{g}_U$ and $\omega : \mathfrak{g}_U \times \mathfrak{g}_U \rightarrow \mathbb{R}$ as follows:

$$[x, y] = R_y^*x - R_x^*y, \quad Jx = ix, \quad \omega(x, y) = \operatorname{Im}(H(x, y)),$$

for all $x, y \in U$.

The pair $(\mathfrak{g}_U, [\cdot, \cdot])$ is a real Lie algebra and (ω, J) provides an abelian pseudo-Kähler structure on \mathfrak{g}_U .

Proof. First note that $R_{\alpha x}^* = \overline{\alpha} R_x^*$ and, obviously, $R_{x+y}^* = R_x^* + R_y^*$ for all $x, y \in U$ and $\alpha \in \mathbb{C}$. This shows that the bracket is \mathbb{R} -bilinear. Further, Jacobi identity follows immediately from the equalities

$$R_{[x,y]}^* = R_y R_x^* - R_x R_y^*, \quad R_x^*[y, z] = R_{zx}^* y - R_{yx}^* z,$$

which can be easily derived from the condition given in equation (1) and the definition of the bracket.

Since H is non-degenerate so must be ω because for every $x, y \in U$ one has

$$H(x, y) = \omega(ix, y) + i\omega(x, y).$$

Further, ω is obviously skew-symmetric, because H is hermitian, and for all $x, y, z \in U$ we get

$$\begin{aligned} & H([x, y], z) + H([y, z], x) + H([z, x], y) = \\ & H(R_y^* x - R_x^* y, z) + H(R_z^* y - R_y^* z, x) + H(R_x^* z - R_z^* x, y) = \\ & H(x, yz) - H(y, xz) + H(y, zx) - H(z, yx) + H(z, xy) - H(x, zy) = 0, \end{aligned}$$

and, therefore, its imaginary part also vanishes, which proves that ω is a symplectic form.

Finally, it is clear that J turns to be an abelian complex structure on \mathfrak{g} and that $\omega(Jx, Jy) = \omega(x, y)$. Thus $(\mathfrak{g}, \omega, J)$ is pseudo-Kähler. \square

Remark 3.1 The following facts concerning the pseudo-Kähler Lie algebra constructed above should be noticed:

- (1) The pseudo-Kähler metric is given by $g(x, y) = \omega(Jx, y) = \operatorname{Re}(H(x, y))$ for all $x, y \in \mathfrak{g}_U$.
- (2) The left-symmetric product defined by ω on \mathfrak{g}_U is as follows:

$$x \cdot y = (R_y + R_y^*)x, \quad x, y \in \mathfrak{g}_U.$$

- (3) It is straightforward to show that when $U = U_1 \oplus U_2$ is a direct sum of H -orthogonal ideals then \mathfrak{g}_U is also the ω -orthogonal direct sum of the ideals \mathfrak{g}_{U_1} and \mathfrak{g}_{U_2} .

The following result shows that every abelian pseudo-Kähler Lie algebra of type $\mathfrak{aff}(\mathcal{A})$ can be viewed as the Lie algebra associated to a compatible pair (U, H) and provides a necessary and sufficient condition to recognize such algebras among those constructed by APK-compatible pairs.

Proposition 3.2 *If (\mathcal{A}, B) is a symmetric associative commutative algebra then $\mathfrak{aff}(\mathcal{A})$ with the standard pseudo-Kähler structure is holomorphically symplectomorphic to the abelian pseudo-Kähler Lie algebra constructed by the APK-compatible pair (U, H) defined by the complexification $U = \mathcal{A}^{\mathbb{C}}$ and the hermitian form given by*

$$H(a + ib, a' + ib') = -4B(a, a') - 4B(b, b') - i4(B(b, a') - B(a, b'))$$

for all $a, b, a', b' \in \mathcal{A}$.

Conversely, if for an abelian pseudo-Kähler Lie algebra \mathfrak{g}_U defined by an APK-compatible pair (U, H) there exists a semilinear involution $\sigma : U \rightarrow U$ such that

$$\sigma(xy) = \sigma(x)\sigma(y), \quad H(\sigma(x), \sigma(y)) = \overline{H(x, y)}, \quad H(xy, z) = H(x, \sigma(y)z)$$

hold for all $x, y, z \in U$, then \mathfrak{g}_U is holomorphically symplectomorphic to $\mathfrak{aff}(\mathcal{A})$ with its standard pseudo-Kähler structure, where \mathcal{A} is the real form of U defined by σ endowed with a bilinear form B obtained, up to constant, by the restriction of H to \mathcal{A} .

Proof. Suppose first that (\mathcal{A}, B) is a symmetric associative commutative algebra and define H on $U = \mathcal{A}^\mathbb{C}$ as in the statement. Notice that H is non-degenerate if and only if B is so. A simple calculation shows that $R_{a+ib}^* = R_{a-ib}$ for all $a, b \in \mathcal{A}$. Thus, the condition (1) follows immediately from the associativity and commutativity of U . If $\psi : \mathfrak{aff}(\mathcal{A}) \rightarrow \mathfrak{g}_U$ is defined by $\psi(a, b) = -(a + ib)/2$, one easily sees that ψ commutes with the complex structures and that $\text{Im}H(\psi(a, b), \psi(a', b')) = B(a, b') - B(b, a')$ for all $a, b, a', b' \in \mathcal{A}$. Moreover, ψ is an isomorphism of real Lie algebras since it is clearly \mathbb{R} -linear and for $a, b, a', b' \in \mathcal{A}$ we have

$$\begin{aligned} [\psi(a, b), \psi(a', b')] &= \frac{1}{4}[a + ib, a' + ib'] = \frac{1}{4}R_{a'+ib'}^*(a + ib) - \frac{1}{4}R_{a+ib}^*(a' + ib') \\ &= \frac{1}{4}R_{a'-ib'}(a + ib) - \frac{1}{4}R_{a-ib}(a' + ib') = \frac{i}{2}(a'b - b'a) = \psi(0, ab' - ba') \\ &= \psi([(a, b), (a', b')]). \end{aligned}$$

Conversely, if (U, H) is an APK-compatible algebra admitting a semilinear involution σ verifying the properties given above, then U admits the associative commutative real form $\mathcal{A} = \{x + \sigma(x); x \in U\}$. For $x, y, z \in U$ we have

$$\begin{aligned} H(x + \sigma(x), y + \sigma(y)) &= 2 \text{Re}(H(x, y) + H(\sigma(x), y)) \\ H((x + \sigma(x))(y + \sigma(y)), z + \sigma(z)) &= H(x + \sigma(x), (y + \sigma(y))(z + \sigma(z))). \end{aligned}$$

From the first identity we get that the form B defined on \mathcal{A} by $B(a, b) = -\frac{1}{4}H(a, b)$ is a real-valued symmetric bilinear form on \mathcal{A} and, from the second, that $B(ab, c) = B(a, bc)$ for $a, b, c \in \mathcal{A}$. Therefore, (\mathcal{A}, B) is a symmetric associative commutative algebra and, according to the first part of the proposition, $\mathfrak{aff}(\mathcal{A})$ is holomorphically symplectomorphic to the Lie algebra constructed with the APK-compatible pair $(\mathcal{A}^\mathbb{C}, \tilde{H})$ where $\tilde{H}(a + ib, a' + ib') = -4B(a, a') - 4B(b, b') - i4(B(b, a') - B(a, b'))$ for all $a, b, a', b' \in \mathcal{A}$. Since $U = \mathcal{A}^\mathbb{C}$, we only need to prove that $\tilde{H} = H$. But this follows at once since both \tilde{H} and H are hermitian and coincide on the real form \mathcal{A} of U . \square

Lemma 3.3 *If (U, H) is an APK-compatible pair and $\text{ann}(U)$ denotes the annihilator of U , then*

$$(R_x^* R_z - R_z R_x^*)y \in \text{ann}(U)$$

for all $x, y, z \in U$.

Proof. Let us consider $u, x, y, z \in U$. Since the product in U is associative and commutative, we have

$$u(R_x^*(zy) - zR_x^*y) = uR_x^*(zy) - uzR_x^*y = zyR_x^*u - z(uR_x^*y) = yzR_x^*u - zyR_x^*u = 0,$$

which proves that $R_x^*(zy) - zR_x^*y$ is in the annihilator. \square

Lemma 3.4 *Let (U, H) be an APK-compatible pair and let $\text{ann}(U)$ denote the annihilator of U and R_U^*U the linear \mathbb{C} -span of all the elements R_x^*y with $x, y \in U$. The following holds:*

- (a) *The vector space R_U^*U is the H -orthogonal subspace of $\text{ann}(U)$ and it is an ideal of the associative commutative algebra U .*
- (b) *If $\text{ann}(U) = \{0\}$ then $U = U^2 = R_U^*U$.*

Proof. It is clear that $H(R_x^*y, x_0) = 0$ for all $x, y \in U$ if and only if $H(y, xx_0) = 0$, this is to say, $x_0 \in \text{ann}(U)$. In order to prove that R_U^*U is an ideal, take $x, y, z \in U$ and $x_0 \in \text{ann}(U)$. We then have

$$H(xR_y^*z, x_0) = H(z, yR_x^*x_0) = H(z, x_0R_x^*y) = 0,$$

which proves that $xR_y^*z \in \text{ann}(U)^\perp = R_U^*U$. This completes the proof of (a).

Now, if $\text{ann}(U) = \{0\}$ then $R_U^*U = \text{ann}(U)^\perp = U$ and hence it only remains to see that it coincides with U^2 . Let us take an element $x \in (U^2)^\perp$. For all $y, z, t \in U$ we then have

$$H(t, xR_y^*z) = H(t, zR_y^*x) = H(yR_z^*t, x) = 0.$$

This clearly implies that $xR_y^*z = 0$ for all $y, z \in U$ but, since $U = R_U^*U$, one obtains $x \in \text{ann}(U) = \{0\}$. Hence, the H -orthogonal subspace of U^2 is null, which shows that $U = U^2$. \square

Proposition 3.5 *Let (U, H) be an APK-compatible pair and $(\mathfrak{g}_U, \omega, J)$ the corresponding pseudo-Kähler Lie algebra with abelian complex structure.*

If $\text{ann}(U) = \{0\}$, then $(\mathfrak{g}_U, \omega, J)$ is holomorphically symplectomorphic to $\mathfrak{aff}(\mathcal{A})$ for some real symmetric associative commutative algebra (\mathcal{A}, B) endowed with the standard pseudo-Kähler structure.

Proof. If $\text{ann}(U) = \{0\}$ we have, according to the lemma above, that $U = R_U^*U$. We can then define a map $\sigma : U \rightarrow U$ by $\sigma(R_x^*y) = R_y^*x$. Note that σ is well defined since if $R_x^*y = R_z^*t$ then for all $u, v \in U$ we get

$$H(R_y^*x - R_t^*z, R_u^*v) = H(R_y^*x, R_u^*v) - H(R_t^*z, R_u^*v) = H(R_v^*u, R_x^*y) - H(R_v^*u, R_z^*t) = 0.$$

It is obvious that $\sigma^2 = \text{id}_U$ and that

$$\sigma(\alpha R_x^*y) = \sigma(R_x^*(\alpha y)) = R_{\alpha y}^*x = \overline{\alpha} R_y^*x = \overline{\alpha} \sigma(R_x^*y).$$

Further, using Lemma 3.3, we get that $R_x^*(yz) = z(R_x^*y)$ holds for all $x, y, z \in U$ and, therefore,

$$\sigma((R_x^*y)(R_z^*t)) = \sigma(tR_z^*(R_x^*y)) = \sigma(R_{zx}^*(yt)) = R_{yt}^*(xz) = (R_y^*x)(R_t^*z) = \sigma(R_x^*y)\sigma(R_z^*t).$$

Bearing in mind Proposition 3.2 it only remains to show that the identities $H(\sigma(x), \sigma(y)) = \overline{H(x, y)}$ and $H(xy, z) = H(x, \sigma(y)z)$ are verified for all $x, y, z \in U$. But this follows at once because from equation (1) we have

$$\begin{aligned} H(R_u^*v, R_x^*y) &= H(R_y^*x, R_v^*u) = \overline{H(R_v^*u, R_y^*x)} \\ H(xR_u^*v, z) &= H(R_u^*v, R_x^*z) = H(R_u^*x, R_v^*z) = H(x, uR_v^*z) = H(x, zR_v^*u), \end{aligned}$$

which yields the desired identity. \square

In the cases where $\text{ann}(U) \cap (\text{ann}(U))^\perp = \{0\}$ something similar occurs, as we prove in the following proposition which also clarifies the structure of the corresponding real symmetric associative commutative algebra.

Proposition 3.6 *Let (U, H) be an APK-compatible pair. If $\text{ann}(U) \cap (\text{ann}(U))^\perp = \{0\}$ then \mathfrak{g}_U is holomorphically symplectomorphic to an algebra $\mathfrak{aff}(\mathcal{A})$ endowed with the standard pseudo-Kähler structure for a certain symmetric form B .*

Moreover, the algebra \mathcal{A} decomposes as a B -orthogonal sum of ideals

$$\mathcal{A} = \text{ann}(\mathcal{A}) \oplus \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_p$$

where p is the number of simple ideals in the semisimple part of \mathcal{A} and each \mathcal{A}_i is a unital algebra.

Proof. If $\text{ann}(U) \cap (\text{ann}(U))^\perp = \{0\}$ then $U = \text{ann}(U) \oplus (\text{ann}(U))^\perp$, which is an orthogonal sum of ideals. The pseudo-Kähler Lie algebra constructed with the algebra $\text{ann}(U)$ is clearly abelian and, therefore, holomorphically symplectomorphic to $\mathfrak{aff}(\mathcal{A}_0)$ for a real algebra with $\mathcal{A}_0^2 = \{0\}$.

The ideal $\tilde{U} = (\text{ann}(U))^\perp$ obviously verifies $\text{ann}(\tilde{U}) = \{0\}$ and according to Proposition 3.5 $\mathfrak{g}_{\tilde{U}}$ is holomorphically symplectomorphic to $\mathfrak{aff}(\tilde{\mathcal{A}})$ for some real symmetric associative commutative algebra $(\tilde{\mathcal{A}}, \tilde{B})$. From $\text{ann}(\tilde{U}) = \{0\}$ we immediately get that $\text{ann}(\tilde{\mathcal{A}})$ is also null. This implies that $\tilde{\mathcal{A}}$ cannot be nilpotent and hence it has a non trivial semisimple part. Since $\tilde{\mathcal{A}}$ is commutative, its semisimple part is the direct sum of several copies of \mathbb{R} and \mathbb{C} with their usual field structure. Let p be the number of those simple ideals and let us choose on each of them an idempotent element e_i . Consider $\mathcal{A}_i = R_{e_i}\tilde{\mathcal{A}}$. It is clear that \mathcal{A}_i is an ideal of $\tilde{\mathcal{A}}$ for all $i \leq p$ with unity e_i , that $\mathcal{A}_i\mathcal{A}_j = \{0\}$ and that they are mutually orthogonal since when $i \neq j$ we have $B(e_ia, e_jb) = B(a, e_ie_jb) = 0$ for all $a, b \in \tilde{\mathcal{A}}$. To see that $\tilde{\mathcal{A}} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_p$, take the Peirce decomposition [1] $\tilde{\mathcal{A}} = e_1\tilde{\mathcal{A}} \oplus \mathcal{M}_1$ where $\mathcal{M}_1 = \{a \in \tilde{\mathcal{A}} : e_1a = 0\}$. Obviously, $\mathcal{A}_2 + \cdots + \mathcal{A}_p \subset \mathcal{M}_1$ and \mathcal{M}_1 has trivial annihilator. We can proceed with \mathcal{M}_1 in the same way, taking its Pierce decomposition relative to the idempotent e_2 . Repeating successively the same argument, we obtain that $\tilde{\mathcal{A}} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_p \oplus \mathcal{M}_p$ where $\mathcal{M}_p = \{a \in \tilde{\mathcal{A}} : e_i a = 0, \text{ for all } i \leq p\}$ is a nilpotent ideal of $\tilde{\mathcal{A}}$. If we suppose that \mathcal{M}_p is not null, then we may find an element $a \in \text{ann}(\mathcal{M}_p)$, $a \neq 0$. But this would imply $a \in \text{ann}(\tilde{\mathcal{A}})$, a contradiction. Hence $\mathcal{M}_p = \{0\}$ and $\tilde{\mathcal{A}} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_p$.

Finally, if we take the orthogonal sum $\mathcal{A} = \mathcal{A}_0 \oplus \tilde{\mathcal{A}}$ one easily proves that $\text{ann}(\mathcal{A}) = \mathcal{A}_0$ and that $\mathfrak{aff}(\mathcal{A}) = \mathfrak{aff}(\mathcal{A}_0) \oplus \mathfrak{aff}(\tilde{\mathcal{A}})$, and the result follows. \square

Remark 3.2 A well-known result on unitary symmetric algebras implies that and the restriction B_i of B to $\mathcal{A}_i \times \mathcal{A}_i$ is given by $B_i(a, b) = f_i(ab)$ for all $a, b \in \mathcal{A}_i$ where $f_i : \mathcal{A}_i \rightarrow \mathbb{R}$ is a linear form such that $f_i \circ R_a \neq 0$ for all $a \in \mathcal{A}_i$.

The main result of this section shows that every abelian pseudo-Kähler Lie algebra is associated to certain APK-compatible complex associative commutative algebra.

Theorem 3.7 *Let $(\mathfrak{g}, \omega, J)$ be a pseudo-Kähler Lie algebra with abelian complex structure. There exists an APK-compatible pair (U, H) such that \mathfrak{g} and \mathfrak{g}_U are holomorphically symplectomorphic.*

Proof. If \mathfrak{g} is pseudo-Kähler, then $(\mathfrak{g}^\mathbb{C}, \omega^\mathbb{C}, K)$ is a para-Kähler complex Lie algebra for $\omega^\mathbb{C}$ and K defined as in Lemma 2.3. Note that $\mathfrak{g}^{1,0} = \ker(J^\mathbb{C} - \text{id}) = \ker(K + \text{id})$ and, according to [6, Lemma 4.1], it is a complex associative commutative algebra. Let us then consider

$$U = \mathfrak{g}^{1,0} = \{x - iJx : x \in \mathfrak{g}\}.$$

Recall that the associative product on U is nothing but the restriction of the left-symmetric product defined on $\mathfrak{g}^{\mathbb{C}}$ by $\omega^{\mathbb{C}}$, this is to say, for $x, y, z, t \in \mathfrak{g}$ we have

$$\omega^{\mathbb{C}}((x - iJx)(y - iJy), z + it) = -\omega^{\mathbb{C}}(y - iJy, [x - iJx, z + it]).$$

Now we can define the non-degenerate hermitian form $H : U \times U \rightarrow \mathbb{C}$ as follows:

$$H(x - iJx, x' - iJx') = 2i\omega^{\mathbb{C}}(x - iJx, x' + iJx') = -4\omega(x, Jx') + i4\omega(x, x'),$$

for $x, x' \in \mathfrak{g}$. We will now show that $u'R_u^*u'' = u''R_u^*u'$ for all $u, u', u'' \in U$. Let us consider

$$\overline{U} = \mathfrak{g}^{0,1} = \{x + iJx : x \in \mathfrak{g}\}$$

and, for $x, y \in \mathfrak{g}$, let us put $\overline{x + iy} = x - iy$. Since $\omega^{\mathbb{C}}(\overline{\xi_1}, \overline{\xi_2}) = \overline{\omega^{\mathbb{C}}(\xi_1, \xi_2)}$ and $[\overline{\xi_1}, \overline{\xi_2}] = \overline{[\xi_1, \xi_2]}$ for all $\xi_1, \xi_2 \in \mathfrak{g}^{\mathbb{C}}$, for the left-symmetric product defined by $\omega^{\mathbb{C}}$ one also has $\overline{\xi_1 \cdot \xi_2} = \overline{\xi_1} \cdot \overline{\xi_2}$ because

$$\omega^{\mathbb{C}}(\overline{\xi_1 \cdot \xi_2}, \xi) = \overline{\omega^{\mathbb{C}}(\xi_1 \cdot \xi_2, \overline{\xi})} = -\overline{\omega^{\mathbb{C}}(\xi_2, [\xi_1, \overline{\xi}])} = -\omega^{\mathbb{C}}(\overline{\xi_2}, [\overline{\xi_1}, \xi]) = \omega^{\mathbb{C}}(\overline{\xi_1} \cdot \overline{\xi_2}, \xi)$$

holds for all $\xi \in \mathfrak{g}^{\mathbb{C}}$. Further, one easily verifies that $\mathfrak{g}^{\mathbb{C}} = U \oplus \overline{U}$ verifies

$$\omega^{\mathbb{C}}(U, U) = \omega^{\mathbb{C}}(\overline{U}, \overline{U}) = \{0\}, \quad u \cdot \overline{u'} \in U, \quad \overline{u} \cdot u \in \overline{U},$$

for all $u, u' \in U$ (which can also be deduced from the results on para-Kähler algebras given in [6]). Thus, for all $u, u', u'' \in U$, we have

$$H(R_u^*u', u'') = H(u', uu'') = 2i\omega^{\mathbb{C}}(u', \overline{uu''}) = -2i\omega^{\mathbb{C}}([\overline{u}, u'], \overline{u''}) = 2i\omega^{\mathbb{C}}(u' \cdot \overline{u}, \overline{u''}) = H(u' \cdot \overline{u}, u'').$$

Therefore we get that $R_u^*u' = u' \cdot \overline{u}$ for all $u, u' \in U$. Since U is commutative and $(\mathfrak{g}^{\mathbb{C}}, \cdot)$ is left-symmetric we get

$$0 = [u', u''] \cdot \overline{u} = u'(u'' \cdot \overline{u}) - u''(u' \cdot \overline{u}) = u'R_u^*u'' - u''R_u^*u' \quad u, u', u'' \in U.$$

We have, thus, proved that (U, H) is an APK-compatible pair.

To see that \mathfrak{g} and \mathfrak{g}_U are holomorphically symplectomorphic, we will consider the map $\psi : \mathfrak{g} \rightarrow \mathfrak{g}_U$ given by $\psi(x) = \frac{1}{2}(x - iJx)$ for all $x \in \mathfrak{g}$. It is clear that $\psi(Jx) = i\psi(x)$, that $\omega(x, y) = \text{Im}(H(\psi(x), \psi(y)))$ and that ψ is \mathbb{R} -linear. Let us prove that $\psi[x, y] = [\psi(x), \psi(y)]_{\mathfrak{g}_U}$. But, if we take $x, y \in \mathfrak{g}$, then

$$\begin{aligned} 4[\psi(x), \psi(y)]_{\mathfrak{g}_U} &= 4R_{\psi(y)}^*\psi(x) - 4R_{\psi(x)}^*\psi(y) = 4\psi(x) \cdot \overline{\psi(y)} - 4\psi(y) \cdot \overline{\psi(x)} \\ &= (x - iJx) \cdot (y + iJy) - (y - iJy) \cdot (x + iJx) \\ &= x \cdot y + Jx \cdot Jy - iJx \cdot y + ix \cdot Jy - y \cdot x - Jy \cdot Jx + iJy \cdot x - iy \cdot Jx \\ &= [x, y] + [Jx, Jy] - iJ[x, y] - iJ[Jx, Jy] = 2([x, y] - iJ[x, y]) = 4\psi[x, y]. \end{aligned}$$

Since ψ is obviously invertible, it is an isomorphism of real Lie algebras. \square

We shall use that every Lie algebra with abelian pseudo-Kähler structure is of the form \mathfrak{g}_U for some APK-compatible pair (U, H) to see when such a Lie algebra is unimodular. We first prove the following lemma:

Lemma 3.8 *Let (U, H) be an APK-compatible pair and consider $x \in U$. For each $k \in \mathbb{N}$, $k \geq 2$ then the power $(R_x + R_x^*)^k$ is a linear combination of R_x^k , $(R_x^*)^k$, $R_x^r(R_x^*)^{k-r}$ and $(R_x^*)^{k-r}R_x^r$, $1 \leq r \leq k-1$.*

Proof. We will proceed by induction on k . The result is clear for $k = 2$ because

$$(R_x + R_x^*)^2 = R_x^2 + R_x R_x^* + R_x^* R_x + (R_x^*)^2.$$

From Lemma 3.3 we obtain that $R_x R_y^* R_x = R_x^2 R_y^*$ and thus $R_x^* R_y R_x^* = (R_x R_y^* R_x)^* = R_y (R_x^*)^2$. Therefore, when $s, r \geq 1$ we have

$$R_x^* R_x^r (R_x^*)^s = R_x^r (R_x^*)^{s+1}, \quad R_x (R_x^*)^s R_x^r = R_x^{r+1} (R_x^*)^s.$$

Now, we have that $(R_x + R_x^*) R_x^{k-1} = R_x^k + R_x^* R_x^{k-1}$, $(R_x + R_x^*) (R_x^*)^{k-1} = R_x (R_x^*)^{k-1} + (R_x^*)^k$ and for $s, r \geq 1$ we get

$$\begin{aligned} (R_x + R_x^*) R_x^r (R_x^*)^s &= R_x^{r+1} (R_x^*)^s + R_x^* R_x^r (R_x^*)^s = R_x^{r+1} (R_x^*)^s + R_x^r (R_x^*)^{s+1}, \\ (R_x + R_x^*) (R_x^*)^s R_x^r &= R_x (R_x^*)^s R_x^r + (R_x^*)^{s+1} R_x^r = R_x^{r+1} (R_x^*)^s + (R_x^*)^{s+1} R_x^r. \end{aligned}$$

and this shows that if $(R_x + R_x^*)^{k-1}$ verifies the condition, so does $(R_x + R_x^*)^k$. \square

Proposition 3.9 *Let \mathfrak{g} be a pseudo-Kähler Lie algebra with abelian complex structure and (U, H) an APK-compatible pair such that $\mathfrak{g} = \mathfrak{g}_U$. The following conditions are equivalent:*

- (i) U is nilpotent,
- (ii) \mathfrak{g} is nilpotent,
- (iii) \mathfrak{g} is unimodular.

To see that \mathfrak{g} is nilpotent whenever U is so, let us consider $x, y \in U$. Recall that $R_{[x, y]}^* = R_y R_x^* - R_x R_y^*$ and hence we have

$$\text{ad}(x)[x, y] = y R_x^* x - x R_y^* x - R_x^*[x, y] = x R_x^* y - x R_y^* x - R_x^*[x, y] = -(R_x + R_x^*)[x, y],$$

and, therefore, $\text{ad}(x)^{k+1}(y) = (-1)^k (R_x + R_x^*)^k [x, y]$ for all $k \geq 1$. If we have that $U^n = \{0\}$ then the Lemma above implies that $(R_x + R_x^*)^{2n} = 0$ and, thus, $\text{ad}(x)^{2n+1} = 0$, which proves the nilpotency of \mathfrak{g} .

It is obvious that if \mathfrak{g} is nilpotent then it is unimodular and, thus, it only remains to show that for a unimodular \mathfrak{g} we get U nilpotent. Note that the map $G_x = \text{ad}(x) + R_x^*$ is semilinear and, therefore, traceless. This shows that $\text{trace}(\text{ad}(x)) = -\text{trace}(R_x^*) = -\text{trace}(R_x)$. Consequently, if \mathfrak{g} is unimodular, then we have $\text{trace}(R_x) = 0$ for all $x \in U$. But this implies that U is nilpotent since, otherwise, there must exist an idempotent element $e \in U$ and one easily proves using the Peirce decomposition of U that $\text{trace}(R_e) = d$ where d is the real dimension of the ideal eU . \square

4 Curvature of the pseudo-Kähler metric

We will now calculate the curvatures of the pseudo-Riemannian metric of a Lie algebra with abelian pseudo-Kähler structure. We shall give two different approaches: a first one using the left-symmetric product defined by the symplectic form, and a second one based on the description in terms of APK-compatible pairs.

Let us consider a Lie algebra with abelian pseudo-Kähler structure $(\mathfrak{g}, \omega, J)$ and denote by (\mathfrak{g}, \cdot) the left-symmetric algebra structure defined by ω .

Proposition 4.1 *Let $(\mathfrak{g}, J, \omega)$ be a pseudo-Kähler Lie algebra with abelian complex structure and g the pseudo-Kähler metric. The Levi-Civita connection and the Riemannian curvature tensor of g are respectively given for $x, y, z \in \mathfrak{g}$ by*

$$\nabla_x y = -Jy \cdot Jx, \quad \mathbf{R}(x, y)z = -Jz \cdot J[x, y] + (z \cdot Jy) \cdot Jx - (z \cdot Jx) \cdot Jy.$$

If \mathcal{K} denotes the Killing form of \mathfrak{g} , then the Ricci curvature tensor $\text{Ric}(x, y) = \text{trace}(\mathbf{R}(x, -)y)$ is given by

$$\text{Ric}(x, y) = \text{trace}(\text{ad}(Jy \cdot Jx)) - \mathcal{K}(x, y), \quad x, y \in \mathfrak{g}.$$

Proof. For $x, y, z \in \mathfrak{g}$ the Levi-Civita connection is given by

$$\begin{aligned} 2g(\nabla_x y, z) &= g([x, y], z) - g([y, z], x) + g([z, x], y) = \omega(J[x, y], z) - \omega(J[y, z], x) + \omega(J[z, x], y) \\ &= \omega(J[x, y], z) - \omega(z, y \cdot Jx) - \omega(z, x \cdot Jy) = \omega(J[x, y] + y \cdot Jx + x \cdot Jy, z) \\ &= g([x, y] - J(y \cdot Jx) - J(x \cdot Jy), z) \end{aligned}$$

and therefore, since J commutes with right multiplications and $[x, y] = [Jx, Jy]$, we have

$$2\nabla_x y = [Jx, Jy] - Jy \cdot Jx - Jx \cdot Jy = -2Jy \cdot Jx,$$

for all $x, y \in \mathfrak{g}$.

Now, the curvature tensor is given by

$$\begin{aligned} \mathbf{R}(x, y)z &= \nabla_{[x, y]}z - \nabla_x \nabla_y z + \nabla_y \nabla_x z = -Jz \cdot J[x, y] - J(Jz \cdot Jy) \cdot Jx + J(Jz \cdot Jx) \cdot Jy \\ &= -Jz \cdot J[x, y] + (z \cdot Jy) \cdot Jx - (z \cdot Jx) \cdot Jy \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

In order to compute the Ricci form, let \mathcal{L}_x denote the left multiplication by an element $x \in \mathfrak{g}$ with the left-symmetric product induced by ω . Then we have $x' \cdot Jx = (\text{ad}(x)J + \mathcal{L}_{Jx})x'$ for all $x' \in \mathfrak{g}$ and it follows that

$$\mathbf{R}(x, z)y = -\mathcal{L}_{Jy}J\text{ad}(x)z + (\text{ad}(x)J + \mathcal{L}_{Jx})\mathcal{L}_y Jz - \mathcal{L}_{y \cdot Jx}Jz.$$

Recalling that $J\mathcal{L}_y = \mathcal{L}_{Jy}$, we get

$$\begin{aligned} \text{Ric}(x, y) &= -\text{trace}(\mathcal{L}_{Jy}J\text{ad}(x)) + \text{trace}(\text{ad}(x)\mathcal{L}_{Jy}J) + \text{trace}(\mathcal{L}_{Jx}\mathcal{L}_y J) - \text{trace}(\mathcal{L}_{y \cdot Jx}J) \\ &= -\text{trace}(\mathcal{L}_x \mathcal{L}_y) - \text{trace}(\mathcal{L}_{Jy \cdot Jx}). \end{aligned}$$

Now, the result follows immediately from the fact that for all $z \in \mathfrak{g}$, the map \mathcal{L}_z is the adjoint map with respect to ω of $-\text{ad}(z)$ and that a linear map and its ω -adjoint have the same trace. \square

Remark 4.1 Observe that, actually, Lemma 2.2 is equivalent to the parallelism of J with respect to the Levi-Civita connection.

Example 4.1 Let (\mathcal{A}, B) be a real symmetric associative commutative algebra and let us consider on $\text{aff}(\mathcal{A})$ the standard pseudo-Kähler structure. We had already seen that the left-symmetric product defined by the symplectic form was given by $(a, b) \cdot (a', b') = -(aa', ba')$ for all $a, b, a', b' \in \mathcal{A}$. Consequently, the Levi-Civita connection and the Riemannian curvature tensor are

$$\nabla_{(a, b)}(a', b') = (b'b, -a'b), \quad \mathbf{R}((a, b), (a', b'))(a'', b'') = (ab'b'' - ba'b'', ba'a'' - ab'a''), \quad a, b, a', b', a'', b'' \in \mathcal{A}.$$

Note that this clearly implies that the metric is flat if and only if $\mathcal{A}^3 = \{0\}$

Besides, we have that

$$\mathbf{R}((a, b), (a', 0))(a'', b'') = (-R_{bb''}a', R_{ba''}a'), \quad \mathbf{R}((a, b), (0, b'))(a'', b'') = (R_{ab''}b', -R_{aa''}b'),$$

from where it is clear that

$$\text{Ric}((a, b), (a'', b'')) = -\text{trace}(R_{bb''}) - \text{trace}(R_{aa''}).$$

Then, the metric is Ricci flat if and only if $\text{trace}(R_{aa'}) = 0$ for all $a, a' \in \mathcal{A}$. But this only occurs if and only if \mathcal{A} is nilpotent because, otherwise, there exists an idempotent $e \in \mathcal{A}$ and one always has $\text{trace}(R_{ee}) = \dim(e\mathcal{A}) \geq 1$.

Flatness or Ricci flatness are easier to study if one describes the Riemannian and the Ricci curvature in terms of the products of an APK-compatible pair. We have the following:

Proposition 4.2 *Let \mathfrak{g}_U denote the pseudo-Kähler Lie algebra with abelian complex structure defined by an APK-compatible pair (U, H) . The Levi-Civita connection, the Riemannian curvature tensor and the Ricci curvature are given for $x, y \in U$ by*

$$\nabla_x y = (R_x - R_x^*)y, \quad \mathbf{R}(x, y) = 3R_x R_y^* - 3R_y R_x^* + R_x^* R_y - R_y^* R_x, \quad \text{Ric}(x, y) = -2 \text{trace}(R_y R_x^*).$$

Proof. According to (2) in Remark 3.1, and Proposition 4.1, we get that the Levi-Civita connection is

$$\nabla_x y = -Jy \cdot Jx = -R_{ix}iy - R_{ix}^*iy = R_x y - R_x^* y.$$

Thus, $\nabla_{[x, y]} = R_{[x, y]} - R_{[x, y]}^* = 2R_x R_y^* - 2R_y R_x^*$ and we then have

$$\mathbf{R}(x, y) = 2R_x R_y^* - 2R_y R_x^* - (R_x - R_x^*)(R_y - R_y^*) + (R_y - R_y^*)(R_x - R_x^*) = 3R_x R_y^* - 3R_y R_x^* + R_x^* R_y - R_y^* R_x.$$

In order to compute the Ricci curvature, let us consider for each $x \in \mathfrak{g}$ the semilinear map $G_x : U \rightarrow U$ defined by $G_x(y) = R_y^* x$. It follows that

$$\mathbf{R}(x, z)y = (3R_x G_y - 3R_y R_x^* + R_x^* R_y - G_{xy})(z).$$

Since $R_x G_y$ and G_{xy} are semilinear, they are traceless and as a consequence we get

$$\text{Ric}(x, y) = -3\text{trace}(R_y R_x^*) + \text{trace}(R_x R_y^*) = -2\text{trace}(R_y R_x^*),$$

as claimed. \square

Corollary 4.3 *Let (U, H) be an APK-compatible pair. The pseudo-Kähler metric on \mathfrak{g}_U is flat if and only if $3R_x R_x^* = R_x^* R_x$ for all $x \in U$.*

Proof. If $3R_x R_x^* - R_x^* R_x = 0$ holds for all $x \in U$ then we have

$$\begin{aligned} 0 &= 3R_{x+y} R_{x+y}^* - R_{x+y}^* R_{x+y} = 3R_x R_y^* + 3R_y R_x^* - R_x^* R_y - R_y^* R_x, \\ 0 &= 3R_{x+iy} R_{x+iy}^* - R_{x+iy}^* R_{x+iy} = -3iR_x R_y^* + 3iR_y R_x^* - iR_x^* R_y + iR_y^* R_x, \end{aligned}$$

which imply that $3R_x R_y^* = R_y^* R_x$ for $x, y \in U$, and one obviously deduces that $\mathbf{R}(x, y) = 0$.

Conversely, if the metric is flat we have

$$0 = R(x, ix) = -6iR_x R_x^* + 2iR_x^* R_x$$

which gives the desired identity. \square

Proposition 4.4 *Let (U, H) be an APK-compatible pair and $(\mathfrak{g}_U, \omega, J)$ the associated pseudo-Kähler Lie algebra. Let us consider the following conditions:*

(c1) *The pseudo-Kähler metric is flat.*

(c2) *The left-symmetric product defined by ω is, actually, associative.*

(c3) *The derived ideal $[\mathfrak{g}, \mathfrak{g}]$ is contained in $\text{ann}(U)$.*

If one of the conditions above is fulfilled, then the other two are equivalent.

Proof. Recall that, as in the proof of the corollary above, the metric is flat if and only if $3R_x R_y^* = R_y^* R_x$ for all $x, y \in U$. On the other hand, the left-symmetric product is associative if and only if

$$0 = R_{x \cdot y} + R_{x \cdot y}^* - (R_y + R_y^*)(R_x + R_x^*) = R_x R_y^* - R_y^* R_x, \quad x, y \in U.$$

Finally, $[\mathfrak{g}, \mathfrak{g}] \subset \text{ann}(U)$ is equivalent to $R_x R_y^* = 0$ for all $x, y \in U$ because we have

$$\begin{aligned} z[x, y] &= zR_y^* x - zR_x^* y = (R_x R_y^* - R_y^* R_x)z \\ z[ix, y] &= izR_y^* x + izR_x^* y = i(R_x R_y^* + R_y^* R_x)z \end{aligned}$$

for all $x, y, z \in U$. Now, it suffices to realize that the combination of two of these conditions imply $R_x R_y^* = R_y^* R_x = 0$ and, then, the remaining condition is also verified. \square

Remark 4.2 A left-symmetric algebra (\mathfrak{A}, \cdot) is called a *Novikov algebra* if $(a \cdot b) \cdot c = (a \cdot c) \cdot b$ is also verified for all $a, b, c \in \mathfrak{A}$ [5],[10]. If \mathfrak{g} is a pseudo-Kähler Lie algebra constructed with an APK-compatible pair (U, H) and (\mathfrak{g}, \cdot) denotes the left-symmetric algebra defined by the symplectic form then we have

$$(x \cdot y) \cdot z - (x \cdot z) \cdot y = (R_z + R_z^*)(R_y + R_y^*)x - (R_y + R_y^*)(R_z + R_z^*)x = (R_z R_y^* + R_z^* R_y - R_y R_z^* - R_y^* R_z)x$$

and, accordingly,

$$(x \cdot y) \cdot iz - (x \cdot iz) \cdot y = (iR_z R_y^* - iR_z^* R_y + iR_y R_z^* - iR_y^* R_z)x.$$

From these two equations we immediately get that (\mathfrak{g}, \cdot) is Novikov if and only if $R_z R_y^* = R_y^* R_z$. This proves that, for our algebras, the Novikov condition is equivalent to the associativity of the left-symmetric product.

Corollary 4.5 *If a pseudo-Kähler algebra with abelian complex structure $(\mathfrak{g}, \omega, J)$ is 2-step nilpotent, then the pseudo-Kähler metric is flat if and only if the left-symmetric product defined by ω is associative.*

Proof. Let (U, H) be an APK-compatible pair for \mathfrak{g} . A simple computation shows that for all $x, y, z \in U$ one has $[z, [x, y]] = -(R_z + R_z^*)[x, y]$ and, therefore, if \mathfrak{g} is 2-nilpotent, then for $x, y, z \in U$ one has

$$0 = [Jz, [x, y]] + J[z, [x, y]] = -(iR_z - iR_z^*)[x, y] - i(R_z + R_z^*)[x, y] = -2iz[x, y],$$

which shows that $[\mathfrak{g}, \mathfrak{g}]$ lies in the annihilator of U and the result is just a consequence of the proposition above. \square

The following proposition, in which we give necessary and sufficient conditions on the APK-compatible pair to yield a 2-step nilpotent Lie algebra and to assure, in such case, that the metric is flat, clarifies the conditions of the Corollary.

Proposition 4.6 *Let (U, H) be an APK-compatible pair. The Lie algebra \mathfrak{g}_U is 2-step nilpotent if and only if $U^3 = \{0\}$ and $(\text{ann}(U))^\perp \subset \text{ann}(U)$. If this is the case, then the pseudo-Kähler metric is flat if and only if $U^2 \subset (U^2)^\perp$.*

Proof. As we had seen in the proofs of Proposition 4.4 and Corollary 4.5, when \mathfrak{g}_U is 2-step nilpotent one has $R_x R_y^* = 0$ for all $x, y \in U$. But this is equivalent to $R_U^* U \subset (R_U^* U)^\perp$ and, since $R_U^* U = (\text{ann}(U))^\perp$, it follows that $(\text{ann}(U))^\perp \subset \text{ann}(U)$. Recalling that we also had $[z, [x, y]] = -(R_z + R_z^*)[x, y]$ for all $x, y, z \in U$, we arrive at

$$0 = i[z, [x, y]] + [z, [x, iy]] = -iR_z^* R_y^* x + iR_z^* R_x^* y - R_z^* R_{iy}^* x + R_z^* R_x^* iy = 2iR_z^* R_x^* y$$

and, thus, we get $H(xzt, y) = H(t, R_z^* R_x^* y) = 0$, showing that $A^3 = \{0\}$. The converse follows at once from the identity $[z, [x, y]] = -(R_z + R_z^*)[x, y]$.

Since $R_x R_y^* = 0$ for all $x, y \in U$, the flatness of the pseudo-Kähler metric is equivalent to $R_x^* R_y = 0$ for all $x, y \in U$. But this is nothing but the condition $H(U^2, U^2) = \{0\}$. \square

Example 4.2 We had seen that if \mathcal{A} is a nilpotent symmetric associative commutative algebra such that $\mathcal{A}^3 \neq \{0\}$, then the standard pseudo-metric on $\mathfrak{aff}(\mathcal{A})$ is not flat. But one can also construct non-flat 2-step nilpotent algebras. For example, let us consider the pair (U, H) where the associative algebra U is the \mathbb{C} -span of $\{u_0, u_1, u_2, u_3, u_4\}$ with the non-trivial products $u_4 u_3 = u_3 u_4 = u_4 u_4 = u_2$ and the hermitian form such that $H(u_j, u_k) = 1$ if $j + k = 4$ and $H(u_j, u_k) = 0$ whenever $j + k \neq 4$. Obviously, $U^3 = \{0\}$ and one has

$$R_{u_3}^* u_2 = u_0, \quad R_{u_4}^* u_2 = u_0 + u_1, \quad R_{u_3}^* u_j = R_{u_4}^* u_j = 0, \quad j \neq 2.$$

Therefore, $R_x R_y^* = 0$ for all $x, y \in U$. This proves that (U, H) is APK-compatible and that the Lie algebra \mathfrak{g}_U is 2-step nilpotent. However, one has $H(U^2, U^2) \neq 0$, which shows that the metric cannot be flat.

Although there exist nilpotent Lie algebras with non flat abelian pseudo-Kähler structures, all of them are Ricci flat as it was shown in a more general context in [14, Lemma 6.3] and can be directly proved:

Corollary 4.7 *Every unimodular pseudo-Kähler Lie algebra with abelian complex structure is Ricci flat.*

Proof. We had seen that for pseudo-Kähler Lie algebras with abelian complex structures unimodularity and nilpotency are equivalent. Further, this means that if (U, H) is an associated APK-compatible pair, then U is nilpotent. This clearly implies that R_x is nilpotent, hence traceless, for all $x \in U$ and, thus, $\text{Ric}(x, y) = -2 \text{trace}(R_y R_x^*) = -2 \text{trace}(R_{R_x^* y}) = 0$. \square

Example 4.3 Bearing in mind the formula given for the Ricci curvature and the explicit calculation done in the case of the algebras $\mathfrak{aff}(\mathcal{A})$, one could think that Ricci flatness is only possible for nilpotent algebras. However, this is not the case. If we consider the pair (U, H) where U is the complex span of $\{u_0, u_1, u_2, u_3\}$ with the only non trivial products $u_1 u_3 = u_3 u_1 = u_1$, $u_3^2 = u_3$ and the hermitian form defined by $H(u_0, u_3) = H(u_1, u_2) = 1$ and $H(u_i, u_j) = 0$ in the remaining pairs $\{i, j\}$, then one easily sees that U is not nilpotent but $R_x R_y^* = R_x^* R_y = 0$ for all $x, y \in U$. Hence, the pseudo-Kähler metric is flat.

We shall finish this section with a characterization of the non Ricci flat Einstein case. Recall that a pseudo-Riemannian metric g is *Einstein* whenever there exists $\gamma \in \mathbb{R}$ such that $g = \gamma \text{Ric}$.

Proposition 4.8 *If $(\mathfrak{g}, \omega, J)$ is a Lie algebra with abelian pseudo-Kähler structure and the pseudo-Kähler metric is Einstein but not Ricci flat then $\mathfrak{g} = \mathfrak{aff}(\mathcal{A})$, endowed with the standard abelian pseudo-Kähler structure, where \mathcal{A} is a semisimple associative commutative algebra and the symmetric form on \mathcal{A} is (up to a scalar) the trace form of the regular representation.*

Proof. Let us consider an APK-compatible pair (U, H) such that $\mathfrak{g} = \mathfrak{g}_U$. If the metric is Einstein, then $\text{Ric}(x, y) = -2\text{trace}(R_y R_x^*)$ must be non-degenerate and, hence, $\text{ann}(U)$ must vanish. According to Proposition 3.5, \mathfrak{g} is holomorphically symplectomorphic to an algebra $\mathfrak{aff}(\mathcal{A})$ for some symmetric associative commutative algebra (\mathcal{A}, B) . But, in this case, we had seen in the Example 4.1 that $\text{Ric}((a, b), (a', b')) = -\text{trace}(R_{bb'}) - \text{trace}(R_{aa'})$ for all $a, b, a', b' \in \mathcal{A}$. Bearing in mind the construction of the pseudo-Kähler metric, we have for all $a, b \in \mathcal{A}$ that

$$B(a, b) = \omega((a, 0), (0, b)) = g((0, -a), (0, b)) = \gamma \text{Ric}((0, -a), (0, b)) = \gamma \text{trace}(R_{ab}).$$

Since such form is non-degenerate if and only if \mathcal{A} is semisimple, the result follows. \square

5 Inductive construction of pseudo-Kähler Lie algebras with abelian complex structure

In [17] and [13], the authors describe the method of symplectic double extensions by one-dimensional Lie algebras to construct new symplectic Lie algebras from a given one. Actually, they prove that every nilpotent symplectic Lie algebra can be obtained by a series of double extensions starting from the zero algebra. The same constructions (slightly modified) have been used in [6] to give an inductive description of all complex para-Kähler Lie algebras with abelian paracomplex structure. However, the method is not applicable in the pseudo-Kähler case unless one uses double extensions by planes instead of lines.

Since every pseudo-Kähler Lie algebra with abelian complex structure is associated to an APK-compatible pair, we will define an inductive method of construction for such pairs. The method is again based on two extensions of a given associative commutative algebra, in a similar way as it is done in [7] for symplectic associative commutative algebras.

All the associative algebras of this section will be, except if the contrary is explicitly said, complex algebras. We first recall some definitions:

Definition 5.1 Let U be an associative commutative algebra and $\varphi : U \times U \rightarrow \mathbb{C}$ a symmetric bilinear form such that $\varphi(xy, z) = \varphi(x, yz)$ for all $x, y, z \in U$. Then the vector space $U \oplus \mathbb{C}u_0$ with the product

$$(x + \alpha u_0)(x' + \alpha' u_0) = xx' + \varphi(x, x')u_0, \quad x, x' \in U, \alpha, \alpha' \in \mathbb{C},$$

is then an associative commutative algebra which we will call the (one-dimensional) *central extension of U by means of φ* .

Definition 5.2 Let U be an associative commutative algebra, $\delta \in \mathfrak{gl}(U)$, $x_0 \in U$ and let $\mathbb{C}v$ denote a one-dimensional associative algebra. On the vector space $\mathbb{C}v \oplus U$ let us define the product

$$(\alpha v + x)(\beta v + y) = \alpha\beta v^2 + \alpha\beta x_0 + xy + \alpha\delta y + \beta\delta x, \quad \alpha, \beta \in \mathbb{C}, x, y \in U.$$

It can be easily seen that $\mathbb{C}v \oplus U$ is an associative commutative algebra if and only if $\delta(xy) = (\delta x)y$ for all $x, y \in U$ and $\delta^2 - \varepsilon \delta = R_{x_0}$ where $\varepsilon \in \mathbb{C}$ is such that $v^2 = \varepsilon v$. In this case we will say that the algebra $\mathbb{C}v \oplus U$ is the *generalized semidirect product of U and $\mathbb{C}v$ by means of (δ, x_0)* .

Our aim is to prove that, essentially, all APK-compatible pairs may be obtained by a double extension given by a central extension and a generalized semidirect product from another APK-compatible pair. We first give conditions to assure that a double extension of that type on an APK-compatible pair leads to another APK-compatible pair.

Proposition 5.1 *Let (U, H) be an APK-compatible pair, $\varepsilon \in \mathbb{C}$, $D \in \mathfrak{gl}(U)$ such that $D(xy) = (Dx)y$ for all $x, y \in U$ and $D^2 - \varepsilon D = R_{a_0}$ for some $a_0 \in U$ and consider a semilinear mapping $\tau : U \rightarrow U$ such that $H(x, \tau y) = \overline{H(\tau x, y)}$. Suppose that there exist $b_0 \in U$ such that*

$$\tau a_0 = (D^* - \varepsilon \text{id})b_0, \quad \tau^2 = DD^* = R_{b_0}, \quad \tau b_0 = Db_0$$

and that for all $x \in U$ the following conditions hold

$$\tau R_x^* = R_x \tau, \quad \tau R_x = R_x^* \tau, \quad D\tau = \tau D^*, \quad DR_x^* = R_{\tau x}, \quad \tau Dx = R_x^* b_0.$$

Let us construct a new algebra $\tilde{U} = \mathbb{C}v_0 \oplus U \oplus \mathbb{C}u_0$ with a commutative product \star such that u_0 annihilates \tilde{U} , $v_0 \star v_0 = \varepsilon v_0 + a_0 + \alpha_0 u_0$ for some $\alpha_0 \in \mathbb{C}$ and

$$x \star y = xy + H(x, \tau y)u_0, \quad v_0 \star x = Dx + H(x, b_0)u_0$$

hold for all $x, y \in U$. If we define a non-degenerate hermitian form \tilde{H} on \tilde{U} by

$$\tilde{H}(u_0, v_0) = 1, \quad \tilde{H}(u_0, u_0) = \tilde{H}(v_0, v_0) = \tilde{H}(u_0, x) = \tilde{H}(v_0, x) = 0, \quad \tilde{H}(x, y) = H(x, y), \quad x, y \in U,$$

then (\tilde{U}, \tilde{H}) is an APK-compatible pair.

Proof. Let us first show that \tilde{U} is actually an associative commutative algebra constructed by performing a central extension on U and then a semidirect product to the extended algebra. It is straightforward to prove that the map $\varphi : U \times U \rightarrow \mathbb{C}$ given by $\varphi(x, y) = H(x, \tau y)$ is bilinear and symmetric. Moreover, for all $x, y, z \in U$ we have

$$\varphi(xy, z) = H(xy, \tau z) = H(x, R_y^* \tau z) = H(x, \tau(yz)) = \varphi(x, yz).$$

This shows that $U' = U \oplus \mathbb{C}u_0$ is the central extension of U by means of φ . Now, if we consider $\delta \in \mathfrak{gl}(U')$ given by $\delta(x + \alpha u_0) = Dx + H(x, b_0)u_0$ for $x \in U, \alpha \in \mathbb{C}$, then we have that

$$\begin{aligned} \delta(x)y &= D(x)y + H(Dx, \tau y)u_0 = D(xy) + \overline{H(\tau Dx, y)}u_0 = D(xy) + \overline{H(R_x^* b_0, y)}u_0 \\ \delta(xy + H(x, \tau y)u_0) &= D(xy) + H(xy, b_0)u_0 \\ (\delta^2 - \varepsilon \delta)x &= D^2 x - \varepsilon Dx + H(x, D^* b_0 - \varepsilon b_0)u_0 = R_{a_0}x + H(x, \tau a_0)u_0 \end{aligned}$$

is verified for all $x, y \in U$. This shows that δ fulfills the necessary conditions to construct a generalized semidirect product of U' and $\mathbb{C}v_0$. Further, one immediately sees that the product on \tilde{U} is nothing but the product on such generalized semidirect product.

It only remains to prove, thus, that the condition (1) of the definition of an APK-compatible pair is verified. Let us denote by \tilde{R}_ξ the multiplication by $\xi \in \tilde{U}$ in \tilde{U} . It is obvious that $\tilde{R}_{u_0}^* = 0$ since $u_0 \in \text{ann}(\tilde{U})$ and rather long but direct calculations yield:

$$\begin{aligned}\tilde{R}_{v_0}^* v_0 &= b_0 + \overline{\alpha_0} u_0, & \tilde{R}_{v_0}^* x &= D^* x + H(x, a_0) u_0, & \tilde{R}_{v_0}^* u_0 &= \bar{\varepsilon} u_0, \\ \tilde{R}_x^* v_0 &= \tau x + H(b_0, x) u_0, & \tilde{R}_x^* y &= R_x^* y + H(y, D x) u_0, & \tilde{R}_x^* u_0 &= 0,\end{aligned}$$

for all $x, y \in U$. We then have for $x \in U$ that

$$\begin{aligned}v_0 \star \tilde{R}_{v_0}^* x &= v_0 \star D^* x = DD^* x + H(D^* x, b_0) u_0 \\ x \star \tilde{R}_{v_0}^* v_0 &= x \star b_0 = x b_0 + H(x, \tau b_0) u_0\end{aligned}$$

and, therefore, they are equal if and only if $DD^* = R_{b_0}$ and $D b_0 = \tau b_0$. If we now take $x, y \in U$, then we get

$$\begin{aligned}x \star \tilde{R}_{v_0}^* y &= x \star D^* y = x D^* y + H(x, \tau D^* y) u_0 \\ y \star \tilde{R}_{v_0}^* x &= y \star D^* x = y D^* x + H(D^* x, \tau y) u_0.\end{aligned}$$

From the equality $DR_x^* = R_{\tau x}$ we get $R_x D^* = R_{\tau x}^*$ and hence, bearing in mind that $\tau R_x^* = R_x \tau$, we obtain for all $z \in U$,

$$H(x D^* y, z) = H(y, z \tau(x)) = H(R_z^* y, \tau x) = H(x, \tau R_z^* y) = H(x, z \tau(y)) = H(y D^* x, z).$$

Since we also have $D\tau = \tau D^*$, we deduce that $x \star \tilde{R}_{v_0}^* y = y \star \tilde{R}_{v_0}^* x$. From the conditions $DR_x^* = R_{\tau x}$ and $\tau^2 = R_{b_0}$ we easily get that $x \star \tilde{R}_y^* v_0 = v_0 \star \tilde{R}_y^* x$ because

$$\begin{aligned}x \star \tilde{R}_y^* v_0 &= x \star \tau y = x \tau(y) + H(x, \tau^2 y) u_0 \\ v_0 \star \tilde{R}_y^* x &= v_0 \star R_y^* x = DR_y^* x + H(R_y^* x, b_0) u_0.\end{aligned}$$

Finally, when $x, y, z \in U$ we have

$$x \star \tilde{R}_y^* z = x \star R_y^* z = x R_y^* z + H(x, \tau R_y^* z) u_0 = z R_y^* x + H(R_y^* x, \tau z) u_0 = z \star R_y^* x = z \star \tilde{R}_y^* x,$$

because (U, H) is APK-compatible and $\tau R_y^* = R_y \tau$. The result now follows at once since the cases involving u_0 are trivial because u_0 and $\tilde{R}_\xi^* u_0$ are in the annihilator of \tilde{U} for all $\xi \in \tilde{U}$. \square

Definition 5.3 Each pair (\tilde{U}, \tilde{H}) constructed as in the proposition above will be called an *APK double extension* of the pair (U, H) by a one-dimensional associative algebra.

Lemma 5.2 Let (U, H) be an APK-compatible pair. If $\text{ann}(U) \cap (\text{ann}(U))^\perp \neq \{0\}$, then there exists $u_0 \in \text{ann}(U) \cap R_U^* U$, $u_0 \neq 0$ and a semilinear map $\lambda : U \rightarrow \mathbb{C}$ such that $R_x^* u_0 = \lambda(x) u_0$ holds for all $x \in U$.

Proof. First notice that $\text{ann}(U) \cap R_U^* U \neq \{0\}$ since $R_U^* U = (\text{ann}(U))^\perp$. If $z \in \text{ann}(U)$ then, for all $x, y \in U$, we have $y R_x^* z = z R_x^* y = 0$, which proves that $\text{ann}(U) \cap R_U^* U$ is stable by all R_x^* , $x \in U$. Since $R_x^* R_y^* = R_{xy}^*$ and $\alpha R_x^* + \beta R_y^* = R_{\alpha x + \beta y}^*$ for all $\alpha, \beta \in \mathbb{C}$, $x, y \in U$, the set $\mathcal{L} = \{\mathcal{R}_x^* : x \in U\}$, where \mathcal{R}_x^* holds for the restriction of R_x^* to $\text{ann}(U) \cap R_U^* U$, is a complex abelian Lie subalgebra of $\mathfrak{gl}(\text{ann}(U) \cap R_U^* U)$. The existence of u_0 and a function $\lambda : U \rightarrow \mathbb{C}$ verifying the condition is then guaranteed by Lie's theorem and the semilinearity of λ is directly deduced from $\alpha R_x^* + \beta R_y^* = R_{\alpha x + \beta y}^*$. \square

Theorem 5.3 *Every APK-compatible pair (\tilde{U}, \tilde{H}) such that $\text{ann}(\tilde{U}) \cap (\text{ann}(\tilde{U}))^\perp \neq \{0\}$ is an APK double extension of another APK-compatible pair (U, H) by a one-dimensional algebra.*

As a consequence, every APK-compatible pair (\tilde{U}, \tilde{H}) is either the pair corresponding to an algebra of type $\mathfrak{aff}(\mathcal{A})$ with its standard pseudo-Kähler structure or can be obtained by a series of APK double extensions starting from the pair associated to an algebra $\mathfrak{aff}(\mathcal{A})$.

Proof. Let us apply the lemma above and consider $u_0 \in \text{ann}(\tilde{U}) \cap \tilde{R}_U^* \tilde{U}$, $u_0 \neq 0$ such that $\tilde{R}_\xi^* u_0 = \lambda(\xi) u_0$ for all $\xi \in \tilde{U}$. It is clear that $\tilde{H}(u_0, u_0) = 0$ since $\tilde{R}_U^* \tilde{U} = (\text{ann}(\tilde{U}))^\perp$. As \tilde{H} is non degenerate, we may find an element $v_0 \in \tilde{U}$ such that $\tilde{H}(v_0, v_0) = 0$ and $\tilde{H}(u_0, v_0) = 1$. This gives a decomposition of vector spaces $\tilde{U} = \mathbb{C}u_0 \oplus U \oplus \mathbb{C}v_0$ where $U = (\mathbb{C}u_0 + \mathbb{C}v_0)^\perp$. The subset $\mathcal{I} = (\mathbb{C}u_0)^\perp = \mathbb{C}u_0 \oplus U$ is actually an ideal of \tilde{U} because for all $x \in \mathcal{I}$ and $\xi \in \tilde{U}$ one has

$$\tilde{H}(u_0, x \star \xi) = \tilde{H}(\tilde{R}_\xi^* u_0, x) = \tilde{H}(\lambda(\xi) u_0, x) = \lambda(\xi) \tilde{H}(u_0, x) = 0,$$

where \star denotes the product on \tilde{U} . Note that the restriction H of \tilde{H} to $U \times U$ is non degenerate and that U is naturally endowed with a structure of associative commutative algebra if we consider the product given by the projection to U of the product in \mathcal{I} . Moreover, if for all $x, y \in U$ we denote $xy \in U$ the projection of $x \star y$ to U , we have $x \star y = xy + \varphi(x, y) u_0$, where φ is certain symmetric bilinear form on U such that $\varphi(xy, z) = \varphi(x, yz)$. Let $\tau : U \rightarrow U$ be the unique semilinear mapping such $\varphi(x, y) = H(x, \tau y)$ for all $x, y \in U$, which is guaranteed because H is non-degenerate. It can be easily shown that $H(x, \tau y) = \overline{H(\tau x, y)}$ and $\tau R_x = R_x^* \tau$ because φ is symmetric and associative. Recall that $\mathcal{I} = \mathbb{C}u_0 \oplus U$ is an ideal of \tilde{U} and, therefore, if we denote by $D \in \mathfrak{gl}(U)$ the projection on U of the restriction of \tilde{R}_{v_0} to U , we can describe the product in \tilde{U} by $u_0 \in \text{ann}(\tilde{U})$ and

$$v_0 \star v_0 = \varepsilon v_0 + a_0 + \alpha_0 u_0, \quad v_0 \star x = Dx + \phi(x) u_0, \quad x \star y = xy + H(x, \tau y) u_0, \quad x, y \in U$$

for some linear form $\phi : U \rightarrow \mathbb{C}$ and certain $\varepsilon, \alpha_0 \in \mathbb{C}$, $a_0 \in U$. The non degeneracy of H let us find $b_0 \in U$ such that $\phi(x) = H(x, b_0)$ for all $x \in U$.

The associativity of \tilde{U} implies that $D(xy) = (Dx)y$, $\tau Dx = R_x^* b_0$ hold for all $x, y \in U$ and that $\tau a_0 = D^* b_0 - \varepsilon b_0$, $D^2 - \varepsilon D = R_{a_0}$ and, since the product \star is given as the one in Proposition 5.1, one sees by simply reversing the arguments of its proof that the condition $\xi \tilde{R}_\nu^* \xi' = \xi' \tilde{R}_\nu^* \xi$ for $\xi, \xi', \nu \in \tilde{U}$ implies that (U, H) is an APK-compatible pair and, further, that all the conditions on τ, D, a_0 and b_0 to have an APK double extension are fulfilled.

The second part of the statement follows by applying successively the first part. If (\tilde{U}, \tilde{H}) is not the pair corresponding to an algebra $\mathfrak{aff}(\mathcal{A})$ then, according to Proposition 3.6, $\text{ann}(\tilde{U}) \cap (\text{ann}(\tilde{U}))^\perp \neq \{0\}$ and it is an APK double extension of certain pair (U_1, H_1) . The same argument can be then applied to U_1 and so on. \square

Remark 5.1 The following facts are remarkable:

- (1) The process to view an APK-compatible pair as a series of double extensions stops when we arrive at an algebra in which the annihilator does not intersect its orthogonal. This means that the symmetric real associative commutative algebra \mathcal{A} of the second part of the theorem can be taken to be one of the algebras considered in Proposition 3.6. In particular, every nilpotent APK-compatible pair (U, H) such that $U^2 \neq \{0\}$ is obtained by a sequence of APK double extensions starting from an algebra with zero multiplication.

- (2) When \mathfrak{g}_U is the Lie algebra constructed with an APK-compatible pair (U, H) , the signature of the pseudo-Kähler metric is twice the one of H and when (\tilde{U}, \tilde{H}) is an APK double extension of a compatible pair (U, H) then the signature of \tilde{H} is given by $\text{sig}(\tilde{H}) = \text{sig}(H) + 1$. This shows that if the pseudo-Kähler metric is definite positive, then \mathfrak{g} cannot be an APK double extension and must actually be an algebra $\mathfrak{aff}(\mathcal{A})$ for an associative commutative symmetric algebra (\mathcal{A}, B) as those in Proposition 3.6, this is to say, an orthogonal sum $\mathcal{A} = \text{ann}(\mathcal{A}) \oplus \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_p$ where each ideal \mathcal{A}_i is of the form $e_i \mathcal{A}$ for some idempotent e_i . When the Kähler metric of $\mathfrak{aff}(\mathcal{A})$ is definite positive, also B must be definite (negative). If the nilradical \mathcal{N}_i of \mathcal{A}_i is non-zero, $x \in \mathcal{N}_i$ and $k \in \mathbb{N}, k \geq 2$ is such that $x^{2k} = 0, x^{2k-2} \neq 0$ then we have $0 = B(x^{2k}, x^2) = B(x^{k+1}, x^{k+1})$ and hence $x^{k+1} = 0$, which is obviously a contradiction with $x^{2k-2} \neq 0$ if $k \geq 3$ and also if $k = 2$ because $x^3 = 0$ implies $B(x^2, x^2) = B(x^3, x) = 0$. Thus $\mathcal{N}_i = \{0\}$ and then $e_i \mathcal{A}$ must be simple. This proves that each ideal $e_i \mathcal{A}$ is isomorphic to either \mathbb{R} or \mathbb{C} . But a simple calculation shows that \mathbb{C} does not admit a definite form of the type defined in Definition 2.5. Thus, one immediately obtains the well-known result (see, for instance, [4, Theorem 4.1]) that a Kähler Lie algebra with abelian complex structure is an orthogonal sum of several copies of $\mathfrak{aff}(\mathbb{R})$ and an even dimensional abelian algebra.

Acknowledgments: We would like to thank Professor Saïd Benayadi for his multiple and interesting suggestions.

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